OCR

Oxford Cambridge and RSA

Monday 4 October 2021

AS Level Further Mathematics B (MEI)

Y410/01 Core Pure

Worked Solutions

Printed Answer Booklet

Time allowed: 1 hour 15 minutes

You must have:
- Question Paper Y410/01 (inside this document)
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

R red level
- longer questions (6+ marks)
- higher level problem solving
- harder A Level content

A amber level
- Shorter questions (3 - 6 marks)
- low level problem solving
- harder AS/easier A Level content.

G green level
- Short questions (1-3 marks)
- minimal problem solving
- AS/easier A Level content.

E explanation

Q1: Series
Q2: Algebra
Q3: Matrices, vectors
Q4: Matrices
Q5: Proof
Q6: Matrices, transformations
Q7: Complex numbers
Q8: Algebra, complex numbers
Q9: Complex numbers

Grade Boundaries

Grade	A	B	C	D	E	U
Mark / 60	38	31	24	17	10	0

Turn over

G

1 Using standard summation formulae, find $\sum_{r=1}^{n}(r^2-3r)$, giving your answer in fully factorised form. **[3]**

$$\sum_{r=1}^{n} r^2 - 3r = \sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r$$ using $\sum ar^2 + br = a\sum r^2 + b\sum r$

In the formula book, $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

and recall $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$. So

$$= \frac{1}{6}n(n+1)(2n+1) - 3\left(\frac{1}{2}n(n+1)\right)$$

$$= \frac{1}{6}n(n+1)\left[(2n+1) - 9\right]$$

$$= \frac{1}{6}n(n+1)(2n-8)$$

factorise fully

$$= \frac{1}{3}n(n+1)(n-4)$$

G

2 The equation $3x^2-4x+2=0$ has roots α and β.

Find an equation with integer coefficients whose roots are $3-2\alpha$ and $3-2\beta$. **[3]**

So we want an equation, in terms of z, that has roots $3-2\alpha$, $3-2\beta$. So $z = 3-2x$

$$x = \frac{1}{2}(3-z)$$

Now sub this into our equation in x and simplify.

$$3\left(\frac{1}{2}(3-z)\right)^2 - 4\left(\frac{1}{2}(3-z)\right) + 2 = 0$$

$$3(3-z)^2 - 8(3-z) + 8 = 0$$

$\times 4$ so our equation has integer coefficients

$$3(9-6z+z^2) - 24 + 8z + 8 = 0$$

$$3z^2 - 18z + 27 + 8z - 16 = 0$$

$$3z^2 - 10z + 11 = 0$$

G

3 Three planes have the following equations.

$$2x - 3y + z = -3,$$
$$x - 4y + 2z = 1,$$
$$-3x - 2y + 3z = 14.$$

(a) (i) Write the system of equations in matrix form. **[1]**

$$\begin{pmatrix} 2 & -3 & 1 \\ 1 & -4 & 2 \\ -3 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 14 \end{pmatrix}$$

G

(ii) Hence find the point of intersection of the planes. **[2]**

Here we need to multiply both sides by the inverse of the coefficient matrix.

(we find this using the calculator)

$(\times M^{-1})$ $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 8 & -7 & 2 \\ 9 & -9 & 3 \\ 14 & -13 & 5 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \\ 14 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$$

hence point of intersection $= (-1, 2, 5)$

A

(b) In this question you must show detailed reasoning.

Find the acute angle between the planes $2x - 3y + z = -3$ and $x - 4y + 2z = 1$. **[4]**

Recall that the angle between two vectors is given by $\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$.

So we can find the angle between the normal to the planes.

$$\cos\theta = \frac{2(1) - 3(-4) + 1(2)}{\sqrt{2^2 + 3^2 + 1^2}\sqrt{1^2 + 4^2 + 2^2}} = \frac{16}{7\sqrt{6}} = \frac{8\sqrt{6}}{21}$$

so $\theta = \arccos\left(\frac{8\sqrt{6}}{21}\right) = 21.070\ldots = 21.1°$

We also know that the angle between the normals is equal to the angle between the planes.

So the angle between the planes is 21.1°.

A

4 Anika thinks that, for two square matrices $\mathbf{A}$ and $\mathbf{B}$, the inverse of $\mathbf{AB}$ is $\mathbf{A}^{-1}\mathbf{B}^{-1}$. Her attempted proof of this is as follows.

$$\begin{aligned}(\mathbf{AB})(\mathbf{A}^{-1}\mathbf{B}^{-1}) &= \mathbf{A}(\mathbf{BA}^{-1})\mathbf{B}^{-1}\\ &= \mathbf{A}(\mathbf{A}^{-1}\mathbf{B})\mathbf{B}^{-1}\\ &= (\mathbf{AA}^{-1})(\mathbf{BB}^{-1})\\ &= \mathbf{I}\times\mathbf{I}\\ &= \mathbf{I}\end{aligned}$$

Hence $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$

(a) Explain the error in Anika's working. **[2]**

Between line 1 and 2, Anika suggests that $BA^{-1} = A^{-1}B$. This is not correct as matrix multiplication is not commutative.

A

(b) State the correct inverse of the matrix $\mathbf{AB}$ and amend Anika's working to prove this. **[3]**

So the correct statement is $(AB)^{-1} = B^{-1}A^{-1}$ $(*)$

to prove this,

$(AB)(B^{-1}A^{-1})$ (this $= I$ if $(*)$ is true)

$= A\,(BB^{-1})\,A^{-1}$

$= A\,(I)A^{-1}$ → remember $MM^{-1} = I$

$= AA^{-1}$

$= I$

A

5 Prove by induction that $\sum_{r=1}^{n} r \times 2^{r-1} = 1 + (n-1)2^n$ for all positive integers n. **[5]**

Step one : base case

When $n=1$, $\sum_{r=1}^{1} r \times 2^{r-1} = 1 \times 2^{1-1} = 1 \times 2^0 = 1$

$1 + (1-1)2^1 = 1 + 0 \times 2 = 1$ $\therefore$ true for $n=1$.

Step two : assumption

Assume true for $n=k$, so $\sum_{r=1}^{k} r \times 2^{r-1} = 1 + (k-1)2^k$

Step three : inductive step

Using the assumed result for $n=k$,

$$\sum_{r=1}^{k+1} r \times 2^{r-1} = \sum_{r=1}^{k} r \times 2^{r-1} + (k+1) \times 2^{(k+1)-1}$$

$$= 1 + (k-1)2^k + (k+1)2^k$$

$$= 1 + 2^k\left[k-1+k+1\right]$$

$$= 1 + 2k \times 2^k$$

$$= 1 + k \times 2^{k+1}$$

$$= 1 + ((k+1)-1)\,2^{(k+1)}$$

$\therefore$ true for $n=k+1$.

Step four : conclusion

If the result is true for $n=k$, it is true for $n=k+1$. Since it is true for $n=1$, it is true for all positive integer values of n.

A

6 A transformation T of the plane has associated matrix $\mathbf{M} = \begin{pmatrix} 1 & \lambda+1 \\ \lambda-1 & -1 \end{pmatrix}$, where λ is a non-zero constant.

(a) (i) Show that T reverses orientation. **[3]**

First let find det M.

$\det M = 1(-1) - (\lambda+1)(\lambda-1)$

$= -1 - (\lambda^2 - 1)$

$= -\lambda^2$

Orientation is reversed if $\det M < 0$.

Since $\lambda > 0$, $\lambda^2 > 0$. So $\det M < 0$ for all $\lambda > 0$, and hence orientation is always reversed.

G1

(ii) State, in terms of λ, the area scale factor of T. **[1]**

The magnitude of det M represents the area scale factor of T.

So area scale factor of T is λ^2

G1

(b) (i) Show that $\mathbf{M}^2 - \lambda^2\mathbf{I} = \mathbf{0}$. **[2]**

By finding M^2 we can show this to be true.

$M^2 = \begin{pmatrix} 1 & \lambda+1 \\ \lambda-1 & -1 \end{pmatrix}\begin{pmatrix} 1 & \lambda+1 \\ \lambda-1 & -1 \end{pmatrix} = \begin{pmatrix} (\lambda+1)(\lambda-1)+1 & \lambda+1-(\lambda+1) \\ \lambda-1-(\lambda-1) & (\lambda+1)(\lambda-1)+1 \end{pmatrix}$

$= \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix} = \lambda^2 I$

hence $M^2 = \lambda^2 I$

$M^2 - \lambda^2 I = 0 \leftarrow$ required result

A

(ii) Hence specify the transformation equivalent to two applications of T. **[1]**

Two applications of M is M^2. So since $M^2 = \lambda^2 I$, the transformation represented by the matrix $\lambda^2 I$ is the answer.

So enlargement about the origin scale factor λ^2

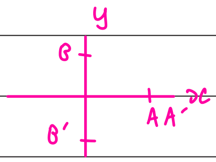
A

(c) In the case where $\lambda = 1$, T is equivalent to a transformation S followed by a reflection in the x-axis.

(i) Determine the matrix associated with S. **[3]**

When $\lambda = 1$, $M = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$

Also, a reflection in the x-ax is is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



Remember that if T is A then B, then T = BA.

So $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$

matrix for S $= \frac{1}{-1}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$\times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1}$

hence associated matrix with S is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

A

(ii) Hence describe the transformation S. **[2]**

Since $(1, 0)$ is invariant and $(0, 1)$ maps to $(2, 1)$, S is a shear.

So transformation S is a shear, x-axis fixed, with $(0, 1)$ mapping to $(2, 1)$

G

7 (a) (i) Find the modulus and argument of z_1, where $z_1 = 1 + \mathrm{i}$. **[2]**

$|z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$ (using $|z| = \sqrt{a^2+b^2}$)

$\arg(z_1) = \arctan\left(\frac{1}{1}\right) = \arctan 1 = \frac{\pi}{4}$ (using $\arg z = \arctan\frac{b}{a}$)

we do not need to adjust this as z is in the 1st Quadrant.

modulus = $\sqrt{2}$

argument = $\frac{\pi}{4}$

G

(ii) Given that $|z_2| = 2$ and $\arg(z_2) = \frac{1}{6}\pi$, express z_2 in $a + b\mathrm{i}$ form, where a and b are exact real numbers. **[2]**

We can write z_2 in mod-arg form, $r(\cos\theta + i\sin\theta)$, and convert this into an $a + bi$ form.

$z_2 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= 2\left(\frac{\sqrt{3}}{2} + i \times \frac{1}{2}\right)$

$= \sqrt{3} + i$

hence $z_2 = \sqrt{3} + i$

R

(b) Using these results, find the exact value of $\sin\frac{5}{12}\pi$, giving the answer in the form $\frac{\sqrt{m}+\sqrt{n}}{p}$, where m, n and p are integers. **[5]**

We can consider $z_1 z_2$ in both $a+bi$ and mod-arg form.

$z_1 z_2 = (1+i)(\sqrt{3}+i) = \sqrt{3} + i + i\sqrt{3} + i^2$ using $i^2 = -1$

$= (-1 + \sqrt{3}) + (1 + \sqrt{3})i$

Also, $|z_1 z_2| = |z_1||z_2|$ and $\arg z_1 z_2 = \arg z_1 + \arg z_2$

So $|z_1 z_2| = |z_1||z_2| = \sqrt{2} \times 2 = 2\sqrt{2}$

$\arg(z_1 z_2) = \arg z_1 + \arg z_2 = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5}{12}\pi$

$\Rightarrow z_1 z_2 = 2\sqrt{2}\left(\cos\frac{5}{12}\pi + i\sin\frac{5}{12}\pi\right)$

Setting these equal to each other,

$2\sqrt{2}\cos\frac{5}{12}\pi + 2\sqrt{2}\, i\sin\frac{5}{12}\pi = (-1+\sqrt{3}) + (1+\sqrt{3})i$

Equating the imaginary parts, $2\sqrt{2}\sin\frac{5}{12}\pi = 1 + \sqrt{3}$

$\sin\frac{5}{12}\pi = \frac{1+\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ (rationilising the denominator)

$\sin\frac{5}{12}\pi = \frac{\sqrt{6}+\sqrt{2}}{4}$

R

8 In this question you must show detailed reasoning.

The equation $x^3 + kx^2 + 15x - 25 = 0$ has roots α, β and $\frac{\alpha}{\beta}$. Given that $\alpha > 0$, find, in any order,

- the roots of the equation,
- the value of k. **[7]**

First we can consider $\sum \alpha\beta$ and $\sum \alpha\beta\gamma$ to find α and β, and thus the roots.

Using $\sum \alpha\beta = \frac{c}{a}$ (product of two roots)

$\alpha\beta + \alpha\left(\frac{\alpha}{\beta}\right) + \beta\left(\frac{\alpha}{\beta}\right) = \frac{15}{1}$

$\alpha\beta + \frac{\alpha^2}{\beta} + \alpha = 15$ $\times \beta$

$\alpha^2 + \alpha\beta^2 + \alpha\beta = 15\beta$ $(*)$

Using $\sum \alpha\beta\gamma = -\frac{d}{a}$ (product of roots)

$\alpha \times \beta \times \frac{\alpha}{\beta} = -\frac{(-25)}{1}$

$\alpha^2 = 25$

$\alpha = \pm 5$ since $\alpha > 0$, $\alpha = 5$

subbing into $(*)$, $5^2 + 5\beta^2 + 5\beta = 15\beta$

$5\beta^2 - 10\beta + 25 = 0$

$\beta^2 - 2\beta + 5 = 0$

$$\beta = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 5}}{2(1)}$$

$= 1 \pm 2i$

So if $\beta = 1+2i$, $\frac{\alpha}{\beta} = \frac{5}{1+2i} \times \frac{1-2i}{1-2i} = 1-2i$

$\beta = 1-2i$, $\frac{\alpha}{\beta} = \frac{5}{1-2i} \times \frac{1+2i}{1+2i} = 1+2i$

hence roots are 5, $1+2i$, $1-2i$

We find k using $\sum \alpha = -\frac{b}{a}$

$5 + 1 + 2i + 1 - 2i = -\frac{k}{1}$

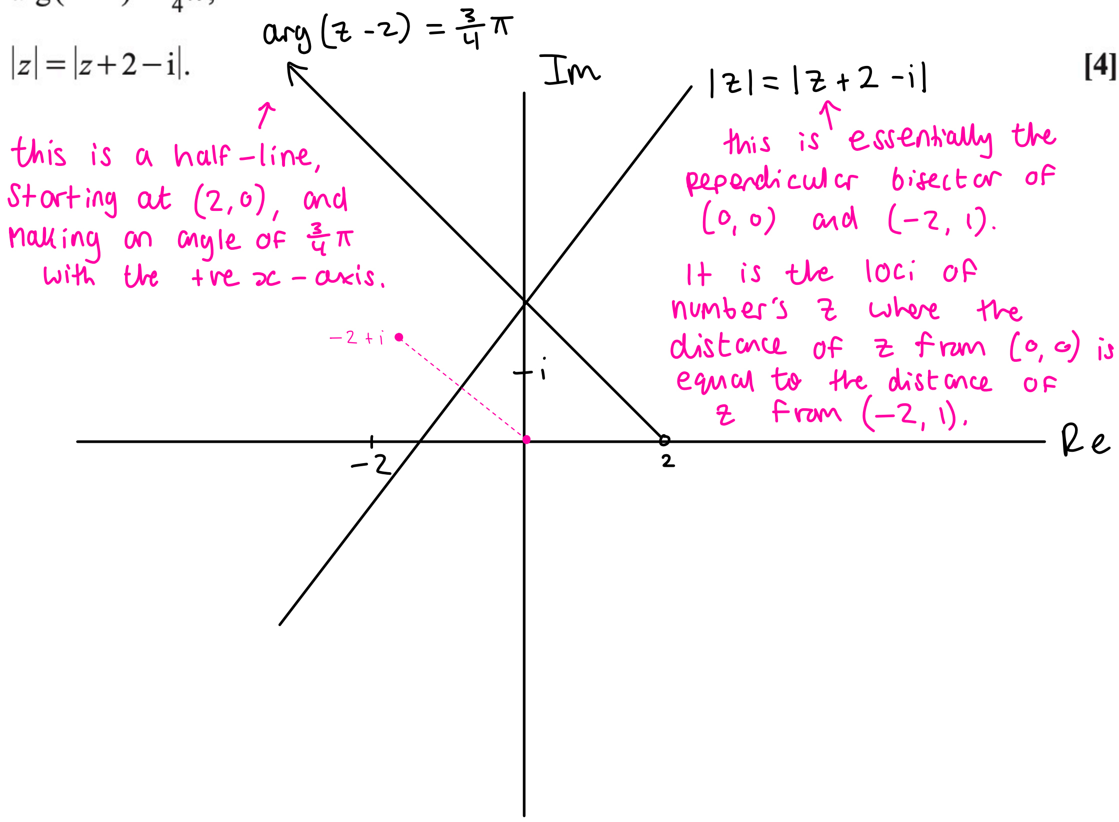
$\Rightarrow 7 - k = 7 \Rightarrow k = -7$

A

9 **(a)** On a single Argand diagram, sketch the loci defined by

- $\arg(z-2) = \frac{3}{4}\pi$,
- $|z| = |z+2-\mathrm{i}|$.

[4]



A

(b) **In this question you must show detailed reasoning.**

The point of intersection of the two loci in part **(a)** represents the complex number w.

Find w, giving your answer in exact form. **[5]**

Here we need to find cartesian equations for both loci and solve them simultaneously to find intersection.

$\arg(z-2) = \frac{3\pi}{4}$: line with $m = -1$, passing through $(2, 0)$

$$y - 0 = -1(x-2)$$

$$y = -x + 2$$

$|z| = |z+2-i|$: perp bisector of $(0, 0)$ and $(-2, 1)$

$$m_1 = -\tfrac{1}{2}, \quad \text{so} \quad m_2 = 2 \qquad \left(m = \frac{\Delta y}{\Delta x}\right)$$

$$\text{midpoint} = \left(-1, \tfrac{1}{2}\right)$$

$$\text{so} \quad y - \tfrac{1}{2} = 2(x+1)$$

$$y = 2x + \tfrac{5}{2}$$

(answer space continued on next page)

9(b)	(continued)
	Solving these simultaneously,
	$2x + \frac{5}{2} = -x + 2$
	$3x = -\frac{1}{2}$
	$x = -\frac{1}{6}$
	and $y = -\left(-\frac{1}{6}\right) + 2 = \frac{13}{6}$
	hence $w = -\frac{1}{6} + \frac{13}{6}i$

ADDITIONAL ANSWER SPACE

If additional space is required, you should use the following lined page(s). The question number(s) must be clearly shown in the margin(s).